HW 2, PROBABILITY I

Let μ be a fixed probability measure on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable, integrable function. Recall the definitions

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}$$

for $p \in [1, \infty)$, and

$$||f||_\infty = ess \sup_{x \in \mathbb{R}} |f(x)|.$$

1. Prove that for a given bounded measurable function $f(x)$, $||f||_p \to_{p\to\infty} ||f||_\infty$.

2. Prove that for a pair of measurable integrable functions f and g,

$$\int_{\infty}^{\infty} |fg| d\mu \leq ||f||_1 ||g||_\infty.$$

3. Prove that $||f||_p \leq ||f||_{p'}$, given that $1 \leq p \leq p' \leq +\infty$, as long as μ is a probability measure. Conclude that if for a positive integer k the $k-$th moment of a random variable X exists, then all of the $m-$th moments for $1 \leq m \leq k$ exist.

4. Show that for every integrable function $f(x)$ on $\mathbb{R}$, the function $g(x) = \int_{\infty}^{x} f(t)dt$ is continuous.

5. Prove Markov's inequality: given a random variable X, an event A and a non-negative function $\varphi : \mathbb{R} \to \mathbb{R}^+$,

$$P(X \in A) \cdot \inf_{y \in A} \varphi(y) \leq \mathbb{E}\varphi(X).$$

6. Find the expectation and the variance for the random variable X with geometric distribution with parameter $p \in (0, 1)$, that is, satisfying for $k = 1, 2, ...$

$$P(X = k) = p(1 - p)^{k-1}.$$

7. Let $X_1, ..., X_n$ be random variables with bounded first moments, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be convex, that is, for all $\lambda \in [0,1]$, $x, y \in \mathbb{R}^n$,

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y).$$

Show that

$$\mathbb{E}\varphi(X_1, ..., X_n) \geq \varphi(\mathbb{E}X_1, ..., \mathbb{E}X_n),$$

provided that $\mathbb{E}|\varphi(X_1, ..., X_n)| < \infty$.

8. Let X be a Poisson random variable with the expectation λ. Find $\mathbb{E}X^3$, $\mathbb{E}X^4$, $\mathbb{E}X^5$.

9. Let X be a random variable with density $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$; prove that $\mathbb{E}X = \mu$ and $Var(X) = \sigma^2$, therefore justifying the name "normal random variable with mean μ and standard deviation σ" for X.

10. Prove that for every random variable X with bounded second moment,

$$\lim_{n\to\infty} \frac{n^2 P(|X| \geq n)}{\mathbb{E}X^2} = 0,$$

thus showing that Chebychev's inequality is not sharp for "long enough tails".
Hint: use monotone convergence theorem.

11. Consider sequence of random variables X_n, $n \geq 1$. Suppose that for $p > 0$ one has

$$\sum_{n=1}^{\infty} \mathbb{E}|X_n|^p < \infty.$$

Prove that $X_n \to 0$ almost everywhere.

12. Fix a probability space $(\Omega, \mathcal{F}, \mu)$. Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables. Let $X_n \to^{\mu} X$ and $Y_n \to^{\mu} Y$. Suppose that $P(X \neq Y) = 0$. Conclude that for every $\epsilon \geq 0$,

$$P(|X_n - Y_n| \geq \epsilon) \to_{n\to\infty} 0.$$